about such concrete matter rather than about general statements of theory that discussion would seem most profitably centered. That there should be such discussion seems to the writer most highly desirable; he is in thorough agreement with Dr. Ruml on this point. In fact, he hopes, with Dr. Ruml, that others may take part in the discussion. He also hopes, with him, that any such discussion may proceed without any of the personal irritation which sometimes so unhappily develops, in the course of differences regarding matters of scientific method. There is surely a need for the clarification of both aims and methods in the field of tests. Such clarification is most decidedly needed, if test work is to continue its healthy growth. If "testing" can thus be brought into its own, the writer believes, as stated in a previous article[9] that the test method will be of great value as a method in pure research. Meanwhile, testing *is* primarily, at present, of a practical nature; and the writer feels that the most healthy and consecutive development, from both the scientific and practical point of view, can most economically be obtained by a vigorous, persistent, and open-minded carrying through of the practical problems to which we are now obligated.

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A NOTE ON THE RÔLE OF MATHEMATICS IN PHYSICS

I

BECAUSE *static* phenomena furnished the model upon which Cartesian geometry was constructed, Leibniz and Newton, especially the latter, were compelled to develop or use a mathematical theory built upon the model of *dynamic* facts as supplied primarily by Da Vinci, Copernicus, Galileo and Kepler. If it is fertile, calculus, as well as any other system of mathematical theory, should bring to light new facts about the physical phenomena as a result of which it was originally constructed. Furthermore the new facts give rise to a new model of physical phenomena. In fact, every new physical fact gives birth to a new model of physical phenomena, upon which mathematicians develop new mathematical theories, which in turn give rise to new generalizations, and if taken up by physicists are the means of discovering new existential facts.

Examination as a Measure of Mental Deterioration," *Journal of Abnormal Psychology*, Vol. XIII, No. 5, December, 1918, pp. 285–294; and S. L. and L. W. Pressey, "The Practical 'Efficiency' of a Group Scale of Intelligence," *Journal of Applied Psychology*, Vol. III, March, 1919, pp. 68–80.

[9] See note 7.

That is, if we become acquainted with a variety of simple experimental facts, we at once begin to connect these facts, if at all possible, into a coherent system; that system becomes an "*ideal*" model in which each and every fact is in some way related to each and every other fact within the model. A new model will lead to a development of a new system of mathematical theory, which, if perfect, may be reduced to a small number of constants and simple equations, and if the latter are developed further, they will lead to new theorems which again may produce new discoveries of physical facts. Therefore it must be said that in their development mathematics and physics depend upon each other to a very large degree. It is physics which may supply to mathematics new models and therefore be indirectly responsible for the development of new mathematical theorems or systems. Mathematics again, because of its generalizations, does not only extend to its own field but also supplies to physics ideas which may directly lead to new discoveries. For example, Newton always began with an analysis of phenomena and then used mathematical synthesis in order to discover new physical facts.

Mathematics plays always an instrumental function in physics, but its rôle is more than merely that of an instrument; it also makes known the *class* to which certain facts belong, but never their *essence.* To believe with Descartes that with matter and motion one could mathematically construct a model of our universe is only an *a priori* idea and not an empirical fact. Even though Cartesians claimed that Descartes's geometry rested upon metaphysical axioms and that the development of these axioms would answer each and every question that physics may raise, some questions could not be answered satisfactorily, consequently a new mathematical system was developed. That does not mean that this new system has taken the place of the old system of theorems, but rather that this new system has taken a place beside the old one, and as a result of this mathematical science was enriched. Descartes defended the mathematical theory of *continuum,* which he based upon a physical model of continuous matter. It was necessary to develop a new mathematical theory when matter was discovered to be discontinuous. The doctrine of molecular discontinuity gave rise to a new model which was instrumental in the development of new mathematical theorems. Once these theorems are deduced, they become entirely independent of the physical theory of molecular discontinuity.

It was Newton and not Descartes who pointed out the function that mathematics must play in discovery. Mathematics must not rest upon axioms which are derived *a priori;* they must be a result of experience, they must be generalizations from models of nature

furnished to us by our experience of facts, whether it be in the street or laboratory. It was Newton who showed by example that we must first analyze facts of experience in order to construct our model, which can be synthesized when mathematically developed. The axioms or definitions are results of analysis (mathematical analysis) which when connected systematically (mathematical synthesis) give rise to new generalizations (due to mathematical deduction) in turn giving rise to new discoveries of physical facts. The truth of these generalizations is directly dependent upon our constant guidance by experience in our procedure.

It would be fallacious to think however that mathematics adapts itself to physics, any more than algebra adapts itself to geometry and gives rise to analytical geometry. The fact is that mathematical physics is the true physics, a fact long recognized by Newton. Mathematical physics is not in any sense opposed to experimental physics; the former is the "*ideal,*" towards which all development of experimental physics should tend. All experiments are means which lead to mathematical explanation; that is actually the true function of mathematics in physics. It is the experiment which gives value or truth to experimental physics, but in its perfect and most useful form it is expressed in mathematical language and consequently the fact of experimental physics becomes the fact of mathematical physics.

Therefore the rôle of mathematics in physics is not that of a discipline independent of facts, and mathematics does *not* give us truth *a priori.* All mathematical physics must begin with facts of experience. To explain fully what may be meant by truth in this connection would necessitate a discussion of the relation of physics to metaphysics, which will be the task of another paper.[1] The exact rôle of mathematics is not to establish mathematical science or mathematical theorems; no more is it to establish *a priori* connections between different facts. These could be known empirically. The rôle of mathematics is to make the connections more easily obtainable and to serve to discover and to express laws, not to prove them, and above all to prove that they have an eternal value.

II

It was Newton's conception of particles or atoms as mathematical points that became so fundamental to all future mechanics, even

[1] We may point out briefly in this connection that it is due to Newton's doctrine of symmetry and not to the Leibnizian principle of sufficient reason that prediction in science is possible, and that truth in physics is intelligible. It is because of the symmetry in the objects of experience themselves that we can rightfully expect a symmetry of effect from a given cause, and not because of the existence of a rational order of principles, which guarantees deduction.

though he thought of them only as mathematical and not as metaphysical entities. That is, Newton, like Galileo before him, thought that the fundamental properties of matter are those only which lend themselves to quantitative treatment. As has been pointed out above, mathematical theory is the result of models constructed from physical facts of experience, and if this mathematical theory is developed, it gives rise to generalizations which give rise to new ideas, when mastered by physicists. These ideas lead to new discoveries and finally change or develop the original model. Consequently, it is natural for the physicist to develop a mathematical theory based upon the quantitative properties of matter, which alone lend themselves to mathematical treatment. These defined properties are derived from experience. A further development of this mathematical theory should lead to new general concepts which in turn give rise to additional or new properties of matter, *e.g.*, system of forces, centripetal force, etc.

Therefore when forces are conceived as applied to each particle of solid, they may also be conceived as applied to a point of the solid called its center of gravity. The summation of these forces in a single point is called by Newton a centripetal force, to which he gave a mathematical expression actually a natural development of Kepler's laws. Newton defines force in terms of algebra, as an expression of an element in movement measured in an element of time. Force and space are in functional relationship, which is scientifically expressed in mathematical terms. Movement is associated with force, and force is a mathematical concept. It was Galileo who gave to Newton the idea of force, and the idea that gravity is a field of such forces, upon which model Newton constructed his mathematical equation. It is very natural for a mathematical physicist to conceive of a body as made up of physical particles between which certain internal actions or forces are constantly working; so that a body is reduced to a system of points and forces. By doing this, the problem of motion and equilibrium resolves itself into an application of the principles of mechanics of the particle. Or the forces of a body, in fact, resolve themselves into a summation of forces, as the force of an immovable center of gravity of a body. Within any such mass, particles or atoms have a geometrical characteristic which is constant, no matter what forces we may apply to them. Consequently it is very natural and fruitful to assume certain rigid relations between particles of any mass, which are regarded as a system of mathematical or physical static relations, and are adaptable to a mathematical treatment, which actually is an *"ideal" simplification* of physical phenomena. But since natural science, *e.g.*, physics or astronomy, deals with large

aggregates of particles, the heterogeneity of relations which *may* actually exist between these particles is negligible, and a single simple mathematical expression of these relations is not only possible but also very desirable because of its utility. As for example when we assert that an ellipsoid with a great number of dimensions can be defined sufficiently by five or six constants, we mean only that our study of mathematical theory, modeled after ellipsoids, can be expressed by means of these five or six constants. Mathematical theory therefore can be reduced to a small number of equations which involve a small number of constants, but it would be folly to say that the actual physical models are so simple in structure. We may notice another example of "*ideal simplification.*" When we express the rate of emission of the α particle, which as a matter of fact it has been actually impossible to see with accuracy, what we really express is the "*mean number,*" and that is really the only thing that we have scientifically arrived at. Here physical facts are only "mean numbers" and these are only the facts accessible to our observation.

We say that A is a function of B if A changes with B, and we can calculate the derivative which will represent this rate of change. We say that this derivative represents the law or rate of change of the body, or in other words, that this derivative stands for the relationship which exists between A and B. But if we carefully observe the change of A and B, whether it be in a physical, chemical, biological or psychological laboratory, and plot the curve representing this functional change, we find that the relationship between A and B is not at all as absolute as the principal concept of calculus has it; in fact the change plotted does not at all represent a curve. What we really have between our X and Y axes, as a fact of actual experience, is a number of points widely distributed, but yet within certain limits representing a continuously progressing curve.[2]

Because the bodies and their motions studied by physicists are apparently so nearly homogeneous, it is therefore possible for them to employ mathematical generalizations of relations in order to express their behavior and nature. It was necessary for Newton to assume that particles are homogeneous, and the models of nature based upon the models of experience dealt with lent themselves to that end. This was necessary, because if the particle (or atoms) were supposed to be heterogeneous no mathematical or any other simple treatment of them would be possible. And furthermore, it was not only necessary to admit the above generalizations, but also that the resultant action of a body would be the direct sum of

[2] I have more fully discussed the above in the *Monist*, for October, 1919.

the actions of all its parts, and that all bodies being made up of similar homogeneous atoms would behave alike. For a generalization contrary to the one stated above would permit any single and simple equation summarizing the behavior of matter, modeled after a large number of diverse facts. That is the reason why the statistical method as employed in science is so fruitful, it always supplies to mathematics the indispensable models, which are constructed on the basis of observable facts which really are the "mean numbers."

Again it is mathematical theory which can bring together under a single generalization the crude facts of isolated experience so different in type as, *e.g.*, the motion of solid bodies, of light, of sound, or of matter.

If the main business of physics (or natural science in general) is to discover and describe the order characteristic of its subject matter, to describe the past and to predict the future with a certain amount of accuracy, then it is legitimate for physics (or science in general) to assume that bodies are functions of their minute units. and that the behavior of a body is the resultant of or a function of the behavior of its units, or that a number of isolated facts can be expressed by a single equation even though this mathematical equation be based upon a model arrived at by a "mean number" statistically derived from observed facts of nature.

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REVIEWS AND ABSTRACTS OF LITERATURE

Implication and Linear Inference. Bernard Bosanquet. London: Macmillan and Co. 1920. Pp. ix + 180.

It is fabled of the hoop-snake that once upon a time he grew discontented with that linear mode of progression so long affected by his ancestors. Thereupon he took the tip of his tail in his mouth, thus producing in his form that endlessness which is the essence of the good infinite and proceeded to roll like a hoop. So alarming was the nimbleness to which he now attained, that he threatened to become the terror of man and beast. Happily for the rest of creation, he chose to attempt the ultimate mountain heights, where the roads are narrow and tortuous. And there it came about, that the very impetuousness of his velocity caused him to skid on an awkward curve, and he tumbled afar into the abyss.

We have before us a volume of hoop-snake logic. Our author is critical of the linear "bead-theory" of deduction, which has been the dominant one in the development of deductive logic, and which